

LATTICE POINT INEQUALITIES FOR CENTERED CONVEX BODIES

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ABSTRACT. We study upper bounds on the number of lattice points for convex bodies having their centroid at the origin. For the family of simplices as well as in the planar case we obtain best possible results. For arbitrary convex bodies we provide an upper bound, which extends the centrally symmetric case and which, in particular, shows that the centroid assumption is indeed much more restrictive than an assumption on the number of interior lattice points even for the class of lattice polytopes.

1. INTRODUCTION

A classical as well as fundamental problem in the Geometry of Numbers is finding bounds on the number of lattice (integral) points $G(K) = \#(K \cap \mathbb{Z}^d)$ of a *convex body* K , i.e., a compact convex set in \mathbb{R}^d , under certain kinds of conditions. Considering the class \mathcal{K}_0^d of all *o-symmetric* convex bodies, i.e., all convex bodies satisfying $K = -K$, this problem has been settled by Minkowski [20, p. 79] under the condition that $\#(\text{int}(K) \cap \mathbb{Z}^d) = 1$, i.e., the origin is the only interior lattice point of K . He proved

$$G(K) \leq 3^d, \quad (1.1)$$

and also gave a better bound of $2^{d+1} - 1$ for strictly convex *o-symmetric* bodies. The equality case in (1.1) has been characterized by Draisma, Nill and McAllister [7]. Furthermore, it was pointed out by Betke et al. [5] that Minkowski's bounds can easily be extended to arbitrary *o-symmetric* convex bodies via the *first successive minimum* $\lambda_1(K)$ of K , which, for latter purposes, we define here for any convex body with $\mathbf{0} \in \text{int}(K)$:

$$\lambda_1(K) = \min\{\lambda \in \mathbb{R}_{>0} : \lambda K \cap \mathbb{Z}^d \neq \{\mathbf{0}\}\}.$$

With this notation Betke et al. [5] showed for **o-symmetric** convex bodies

$$G(K) \leq \left\lfloor \frac{2}{\lambda_1(K)} + 1 \right\rfloor^d, \quad (1.2)$$

where $\lfloor x \rfloor$ is the largest integer not larger than x .

In order to bound the number of lattice points of more general convex bodies one certainly needs restrictions either on the class of convex bodies or on the position of the origin. For instance, even for simplices with one interior lattice point there is no general upper bound as the family of triangles with vertices $(-m, -1), (m, -1)$ and $(0, 1/(m-1))$, $m > 1$, shows.

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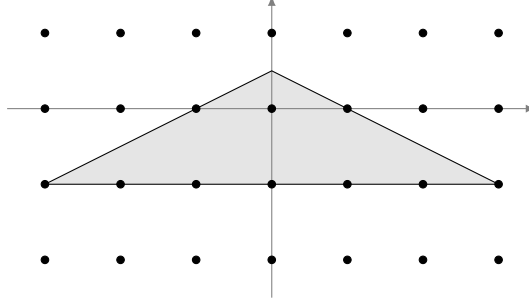


FIGURE 1.

If we are dealing, however, only with lattice polytopes, i.e., all vertices are integral, then there are bounds on the number of lattice points (actually, on the volume) in terms of the (non-zero) number of interior lattice points, see., e.g., Hensley [17], Lagarias & Ziegler [18], Pikhurko [23, 24], Averkov [1].

Even in the case of lattice simplices having only one interior lattice point such an upper bound has to be double exponential in d as shown by Perles, Wills and Zaks [28]. They presented a simplex $S_1^d \subset \mathbb{R}^d$ with a single interior lattice point and

$$G(S_1^d) \geq \frac{2}{6(d-2)!} 2^{2^{d-a}},$$

where $a = 0.5856\dots$ is a constant. Averkov, Krümpelmann and Nill [2] proved that S_1^d has maximum volume among all lattice simplices having one interior lattice point. It remains an open question if a similar result is true considering the number of lattice points instead of the volume as conjectured by Hensley [17].

In this work we show that the number of lattice points of a convex body K with $\text{int}(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$ having its centroid at the origin is (still – cf. (1.1)) at most exponential in the dimension d . The *centroid* or *barycenter* of a Lebesgue measurable set $A \subset \mathbb{R}^d$ having positive Lebesgue measure is defined as

$$c(A) = \frac{1}{\text{vol}(A)} \int_A \mathbf{x} d\mathbf{x},$$

where $\text{vol}(A)$ denotes the *volume*, i.e. the $(d\text{-dimensional})$ Lebesgue measure of A , and $d\mathbf{x}$ integration with respect to the Lebesgue measure. Letting \mathcal{K}_c^d denote the class of all convex bodies K with $c(K) = \mathbf{0}$ we prove the following result.

Proposition 1.1. *Let $K \in \mathcal{K}_c^d$. Then*

$$G(K) < 2^d \left(\frac{2}{\lambda_1(K)} + 1 \right)^d. \quad (1.3)$$

Particularly, if $\lambda_1(K) \geq 1$, i.e., $\text{int}(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$, Proposition 1.1 yields $G(K) < 6^d$.

The bound in Proposition 1.1 is quite likely asymptotically not sharp and we believe that the worst case is attained by d -simplices. More precisely, let $\mathbf{e}_i \in \mathbb{R}^d$ be the i th unit vector, $\mathbf{1} = (1, \dots, 1)$ the all ones vector and let

$\text{conv } A$ be the *convex hull* of a given point set $A \subset \mathbb{R}^d$. Let

$$\begin{aligned} S_d &:= (d+1)\text{conv}\{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d\} - \mathbf{1} \\ &= \{\mathbf{x} \in \mathbb{R}^d : x_i \geq -1, 1 \leq i \leq d, x_1 + x_2 + \dots + x_d \leq 1\}. \end{aligned} \quad (1.4)$$

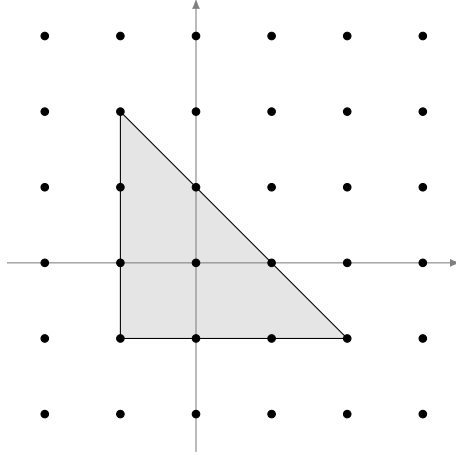


FIGURE 2. S_2

For an integer $m \in \mathbb{N}$ it is easily verified that $G(mS_d) = \binom{d+m(d+1)}{d}$ and we believe that this type of bound is the correct upper bound in Proposition 1.3, i.e.,

Conjecture 1. *Let $K \in \mathcal{K}_c^d$. Then*

$$G(K) \leq \binom{d + \lceil \lambda_1(K)^{-1}(d+1) \rceil}{d}.$$

Observe, compared to (1.3) this bound is asymptotically smaller by a factor of $(e/4)^d$. We will verify this conjecture for arbitrary simplices.

Theorem 1.1. *Let $S \in \mathcal{K}_c^d$ be a d -simplex. Then*

$$G(S) \leq \binom{d + \lceil \lambda_1(S)^{-1}(d+1) \rceil}{d}.$$

Furthermore, for $\lambda_1(S)^{-1} \in \mathbb{N}$ equality holds if and only if S is unimodularly equivalent to $\lambda_1(S)^{-1} S_d$.

Here we say that two sets $A, B \subset \mathbb{R}^d$ are *unimodularly equivalent* if $A = UB + \mathbf{z}$ for a unimodular matrix $U \in \mathbb{Z}^{d \times d}$, i.e., $|\det U| = 1$, and $\mathbf{z} \in \mathbb{Z}^d$.

It is noteworthy that Conjecture 1 is strongly related to a well-known conjecture by Ehrhart.

Conjecture (Ehrhart, [13]). *Let $K \subset \mathbb{R}^d$ be a convex body with $\mathbf{c}(K) = \mathbf{0}$, and $\text{int}(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$. Then*

$$\text{vol}(K) \leq \text{vol}(S_d) = \frac{(d+1)^d}{d!}.$$

In fact, due to the Jordan measurability of convex bodies (cf., e.g., [14, Theorem 7.4]), Conjecture 1 implies Ehrhart's conjecture:

$$\begin{aligned}
\text{vol}(K) &= \lim_{m \rightarrow \infty} m^{-d} \# \left(K \cap \frac{1}{m} \mathbb{Z}^d \right) = \lim_{m \rightarrow \infty} m^{-d} \# (mK \cap \mathbb{Z}^d) \\
&\leq \lim_{m \rightarrow \infty} m^{-d} \binom{d + \left\lceil \lambda_1(mK)^{-1}(d+1) \right\rceil}{d} \\
&= \lim_{m \rightarrow \infty} m^{-d} \binom{d + \left\lceil m\lambda_1(K)^{-1}(d+1) \right\rceil}{d} \\
&\leq \lim_{m \rightarrow \infty} m^{-d} \binom{d + m\lambda_1(K)^{-1}(d+1) + 1}{d} = \lambda_1(K)^{-d} \frac{(d+1)^d}{d!}.
\end{aligned}$$

Hence, if $\text{int}(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$ we have $\lambda_1(K) \geq 1$ and Conjecture 1 gives Ehrhart's conjecture. For recent progress regarding the latter we refer to [4] and [21]. Ehrhart proved his conjecture in the plane [8, 13] and for simplices in any dimension [12]. Observe, that Theorem 1.1 also implies Ehrhart's result for simplices (cf. [16, Proposition 2.15]). The problem is also briefly discussed in [6, p. 147]. It is worth mentioning that S_d has maximal volume among all lattice simplices having the centroid at the origin, see Lemma 5 in [24]. Moreover, every such lattice simplex has also constant Mahler volume [2, Theorem 2.4, Proposition 6.1].

Finally, by using classical results of planar geometry, our last result verifies Conjecture 1 for planar convex bodies whose only lattice point is the origin.

Theorem 1.2. *Let $K \in \mathcal{K}_c^2$ with $G(\text{int}(K)) = 1$. Then*

$$G(K) \leq 10.$$

Furthermore, equality holds if and only if K is unimodularly equivalent to S_2 .

This paper is organised as follows. In Section 2 we will discuss the proof of Proposition 1.1 in detail. The proof of Theorem 1.1 is presented in Section 3. It relies mainly on the fact that a point in a simplex can be described in a unique way by its barycentric coordinates. In Section 4 we will provide the proof of Theorem 1.2, which is based on results due to Ehrhart, Grünbaum, Pick and Scott.

2. THE PROOF OF PROPOSITION 1.1

Considering a convex body K it appears plausible that its centroid $c(K)$ is located deep inside K , i.e., not too close to the boundary of K . In conclusion one would expect that the volume of the intersection of K with its reflection at $c(K)$ cannot be too small with respect to the volume of K . Indeed Milman and Pajor proved the following result on which our proof relies heavily.

Theorem 2.1 ([19, Corollary 3]). *Let $K \in \mathcal{K}_c^d$ with $c(K) = \mathbf{0}$. Then*

$$\text{vol}(K \cap -K) \geq 2^{-d} \text{vol}(K).$$

Proof of Proposition 1.1. Let $K \in \mathcal{K}_c^d$ and for short we set $\lambda_1 = \lambda_1(K)$. First we observe that $\frac{\lambda_1}{2}(K \cap -K)$ is a packing set with respect to the integer lattice, i.e., we have

$$\text{int} \left(\mathbf{u} + \frac{\lambda_1}{2}(K \cap -K) \right) \cap \text{int} \left(\mathbf{v} + \frac{\lambda_1}{2}(K \cap -K) \right) = \emptyset,$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$, $\mathbf{u} \neq \mathbf{v}$; otherwise, by the o -symmetry of $K \cap -K$ we get

$$\begin{aligned} \mathbf{u} - \mathbf{v} &\in \frac{\lambda_1}{2} \text{int}(K \cap -K) + \frac{\lambda_1}{2} \text{int}(K \cap -K) \\ &= \lambda_1 \text{int}(K \cap -K) \subseteq \lambda_1 \text{int}(K), \end{aligned}$$

contradicting the minimality of λ_1 . We also certainly have (cf. Figure 3)

$$(\mathbb{Z}^d \cap K) + \frac{\lambda_1}{2}(K \cap -K) \subseteq K + \frac{\lambda_1}{2}(K \cap -K) \subseteq \left(1 + \frac{\lambda_1}{2}\right) K. \quad (2.1)$$

Hence, in view of Theorem 2.1 we find

$$\begin{aligned} G(K) &\leq \frac{\text{vol} \left(\left(1 + \frac{\lambda_1}{2}\right) K \right)}{\text{vol} \left(\frac{\lambda_1}{2}(K \cap -K) \right)} = \left(\frac{2}{\lambda_1} + 1 \right)^d \frac{\text{vol}(K)}{\text{vol}(K \cap -K)} \\ &\leq 2^d \left(\frac{2}{\lambda_1} + 1 \right)^d. \end{aligned} \quad (2.2)$$

If K is o -symmetric we know by (1.2) that the last inequality is strict, and if K is not symmetric with respect to $\mathbf{0}$ we have a strict inclusion in the last inclusion of (2.1). \square

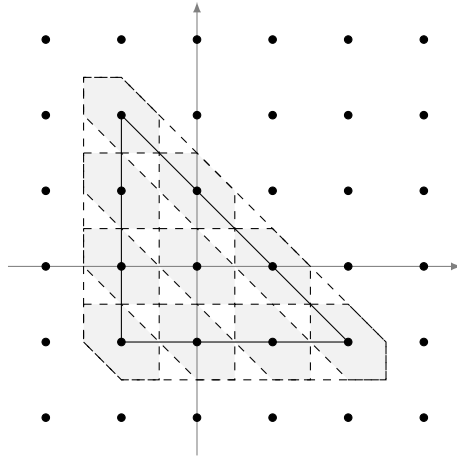


FIGURE 3.

Remark 2.1. In the case that the convex body K is centrally symmetric the proof above gives essentially the result (1.2) since then $K \cap -K = K$ in (2.2). In particular, it also recovers Minkowski's result (1.1) which he proved by a simple residue class argument. Actually, by such an argument

Minkowski also showed that in the case of strictly o -symmetric convex bodies K with $\lambda_1(K) = 1$ the stronger bound holds true

$$G(K) \leq 2^{d+1} - 1.$$

In fact, this bound is even true without symmetry solely under the assumption $\text{int}(K) \cap \mathbb{Z}^d = \{\mathbf{0}\}$.

Proposition 1.1 is certainly not sharp, since for non o -symmetric convex bodies the left-hand side in (2.1) is a proper subset of the right-hand side (see also in Figure 3). In addition, it is unknown whether Corollary 2.1 is sharp itself. In fact, it seems rather plausible that the volume ratio $\text{vol}(K \cap -K) / \text{vol}(K)$ is minimal if K is a simplex for which it is exponentially greater than 2^{-d} , namely (roughly) of order $(2/e)^d$.

For $d = 2$ this problem has been solved by Stewart, c.f. [26]. For recent results regarding this problem see [27].

3. PROOF OF THEOREM 1.1

Let $S = \text{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_{d+1}\} \subset \mathbb{R}^d$ be a d -dimensional simplex, and for $\mathbf{x} \in \mathbb{R}^d$ let $\beta_S(\mathbf{x}) \in \mathbb{R}^{d+1}$ its unique *affine (barycentric)* coordinates with respect to the vertices of $\mathbf{v}_1, \dots, \mathbf{v}_{d+1}$ of S , i.e.,

$$\mathbf{x} = \sum_{i=1}^{d+1} \beta_S(\mathbf{x})_i \mathbf{v}_i \quad \text{and} \quad \sum_{i=1}^{d+1} \beta_S(\mathbf{x})_i = 1.$$

Moreover, we have $\mathbf{x} \in S$ if and only if $\beta_S(\mathbf{x}) \in \mathbb{R}_{\geq 0}^{d+1}$, i.e., all entries are non-negative, as well as $\mathbf{x} \in \text{int}(S)$ if and only if $\beta_S(\mathbf{x}) \in \mathbb{R}_{> 0}^{d+1}$. For \mathbf{x}, \mathbf{y} and $\lambda, \mu \in \mathbb{R}$ we have

$$\beta_S(\mu \mathbf{x} + \lambda \mathbf{y}) = \mu \beta_S(\mathbf{x}) + \lambda \beta_S(\mathbf{y}) + (1 - (\mu + \lambda)) \beta_S(\mathbf{0}).$$

We denote by \mathcal{S}^d the class of all d -dimensional simplices and by \mathcal{S}_c^d the set of all $S \in \mathcal{S}^d$ with $c(S) = \mathbf{0}$. Observe, for $S \in \mathcal{S}_c^d$ we have

$$\beta_S(\mathbf{0})_i = \frac{1}{d+1}, \quad 1 \leq i \leq d+1.$$

Lemma 3.1. *Let $S \in \mathcal{S}_c^d$, and let $\mathbf{u}, \mathbf{w} \in S \cap \mathbb{Z}^d$, $\mathbf{u} \neq \mathbf{w}$. Then there exists an index $k \in \{1, \dots, n+1\}$ with*

$$\beta_S(\mathbf{u})_k - \beta_S(\mathbf{w})_k \geq \lambda_1(S) \frac{1}{d+1}.$$

Proof. Suppose the opposite, i.e., $\beta_S(\mathbf{u})_i - \beta_S(\mathbf{w})_i < \lambda_1/(d+1)$ for all $1 \leq i \leq d+1$, and let $\mathbf{v} = \mathbf{w} - \mathbf{u}$. Then with $\lambda_1 = \lambda_1(S)$ we get

$$\beta_S\left(\frac{1}{\lambda_1} \mathbf{v}\right) = \frac{1}{\lambda_1} (\beta_S(\mathbf{w}) - \beta_S(\mathbf{u})) + \beta_S(\mathbf{0}) > \mathbf{0}.$$

Hence, the non-trivial lattice point \mathbf{v} belongs to $\text{int}(\lambda_1 S)$, and thus contradicting the minimality of λ_1 . \square

Obviously, Lemma 3.1 says, that if two lattice points \mathbf{u} and \mathbf{w} in a given simplex are located close to each other, that is, the difference of their barycentric coordinates is small, then $\mathbf{u} - \mathbf{w}$ will lie inside the simplex as pictured in Figure 4.

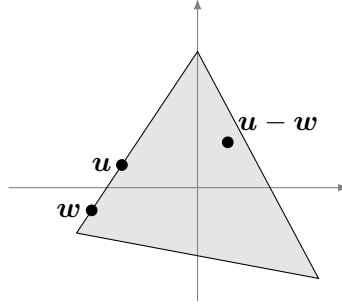


FIGURE 4.

We introduce some more notation. Let

$$B = \left\{ \mathbf{x} \in \mathbb{R}_{\geq 0}^{d+1} : \sum_{i=1}^{d+1} x_i = 1 \right\},$$

which we regard as the d -dimensional simplex in \mathbb{R}^{d+1} of all feasible barycentric coordinates of points contained in a d -dimensional simplex. For a given real number $\rho > 0$ let

$$n(\rho) = \lceil \rho^{-1}(d+1) \rceil \quad (3.1)$$

and

$$R_\rho = \frac{1}{n(\rho)} \left\{ \mathbf{a} \in \mathbb{Z}_{\geq 0}^{d+1} : \sum_{i=1}^{d+1} a_i = n(\rho) \right\}.$$

Note that $\#(R_\rho) = \binom{d+n(\rho)}{d}$ and $R_\rho \subset B$. Let $Z_\rho = \left[0, \frac{1}{n(\rho)}\right)^d \times \mathbb{R} \subset \mathbb{R}^{d+1}$ be the cylinder over the half-open d -dimensional cube of edge length $\frac{1}{n(\rho)}$. For $\mathbf{r} \in R_\rho$ the intersection of $\mathbf{r} + Z_\rho$ with the affine space $\{\mathbf{x} \in \mathbb{R}^{d+1} : \sum x_i = 1\}$ containing B yields a half-open d -dimensional parallelepiped, as depicted in Figure 5.

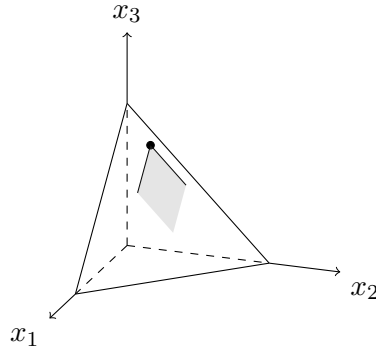


FIGURE 5.

Next we claim

Lemma 3.2. *With the notation above we have $B \subset R_\rho + Z_\rho$.*

Proof. We may write

$$R_\rho + Z_\rho = \bigcup_{\mathbf{r} \in R_\rho} \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : r_i \leq x_i < r_i + \frac{1}{n(\rho)}, i = 1, \dots, d \right\}. \quad (3.2)$$

For a given $\mathbf{x} \in B$ let $\mathbf{r} \in \mathbb{R}^{d+1}$ be defined as

$$\begin{aligned} r_i &= \frac{\lfloor n(\rho)x_i \rfloor}{n(\rho)} \quad \text{for } 1 \leq i \leq d, \\ r_{d+1} &= 1 - \sum_{i=1}^d r_i = \frac{n(\rho) - \sum_{i=1}^d \lfloor n(\rho)x_i \rfloor}{n(\rho)}. \end{aligned} \quad (3.3)$$

Obviously $r_i \geq 0$, $1 \leq i \leq d$, $\sum_{i=1}^d r_i = 1$ and

$$r_{d+1} = \frac{n(\rho) - \sum_{i=1}^d \lfloor n(\rho)x_i \rfloor}{n(\rho)} \geq 1 - \sum_{i=1}^d x_i = x_{d+1} \geq 0.$$

Hence, $\mathbf{r} \in R_\rho$. We also have $r_i \leq x_i$, $1 \leq i \leq d$, as well as

$$x_i - r_i = \frac{n(\rho)x_i - \lfloor n(\rho)x_i \rfloor}{n(\rho)} \leq \frac{1}{n(\rho)}.$$

Thus we have shown $\mathbf{x} \in \mathbf{r} + Z_\rho$. \square

Although not needed for the proof we remark that the union in (3.2) is disjoint.

Proof of Theorem 1.1. Let $S \in \mathcal{S}_c^d$ be a d -simplex and for short we write $\lambda_1 = \lambda_1(S)$. Suppose

$$G(S) > \binom{d + n(\lambda_1)}{d} = \#(R_{\lambda_1}).$$

According to Lemma 3.2 the set of all barycentric coordinates B of points in S is covered by the cylinders $R_{\lambda_1} + Z_{\lambda_1}$. Hence there exist an $\mathbf{r} \in R_{\lambda_1}$ and two lattice points $\mathbf{u}, \mathbf{w} \in S$ such that

$$\beta_S(\mathbf{u}), \beta_S(\mathbf{w}) \in \mathbf{r} + Z_{\lambda_1}.$$

We may assume $\beta_S(\mathbf{u})_{d+1} - \beta_S(\mathbf{w})_{d+1} < 1/n(\lambda_1)$ and due to the definition of Z_{λ_1} we also have $|\beta_S(\mathbf{u})_i - \beta_S(\mathbf{w})_i| < 1/n(\lambda_1)$, $1 \leq i \leq d$. Thus

$$\beta_S(\mathbf{u})_i - \beta_S(\mathbf{w})_i < \frac{1}{n(\lambda_1)} \leq \lambda_1 \frac{1}{d+1}$$

contradicting Lemma 3.1.

Next we discuss the equality case. Let $\lambda_1^{-1} \in \mathbb{N}$ and let $G(S) = \binom{d + n(\lambda_1)}{d}$. We first show that the set $\beta_S(S \cap \mathbb{Z}^d)$, consisting of the barycentric coordinates of all lattice points in S , equals R_{λ_1} . To this end, suppose there exists $\mathbf{x} \in X = \beta_S(S \cap \mathbb{Z}^d) \setminus R_{\lambda_1}$. Thus, there exists an index ℓ such that $x_\ell \neq \frac{k}{n(\lambda_1)}$ for every integer $0 \leq k \leq n(\lambda_1)$. Let \mathbf{x} be such a point in X having maximal x_ℓ . We may assume $\ell \neq d+1$ and define \mathbf{r} as in (3.3),

where $\rho = \lambda_1$. Accordingly it holds $\mathbf{r} \in \beta_S(S \cap \mathbb{Z}^d)$. Since $r_\ell < x_\ell$, we have $r_{d+1} > 0$ and thus $r_{d+1} \geq \frac{1}{n(\lambda_1)}$. Let

$$\begin{aligned} t_i &= r_i \quad \text{for } 1 \leq i \leq d, i \neq \ell, \\ t_\ell &= r_\ell + \frac{1}{n(\lambda_1)} = \frac{\lfloor n(\rho)x_i \rfloor + 1}{n(\lambda_1)}, \\ t_{d+1} &= r_{d+1} - \frac{1}{n(\lambda_1)}. \end{aligned}$$

Then $\mathbf{t} \in R_{\lambda_1}$. Let $\mathbf{v} \in S \cap \mathbb{Z}^d$ be the unique lattice point in S such that $\beta_S(\mathbf{v}) \in \mathbf{t} + Z_{\lambda_1}$. Then $\beta_S(\mathbf{v})_\ell = t_\ell$ by the assumption on \mathbf{x} . We now conclude

$$\begin{aligned} x_i - \beta_S(\mathbf{v})_i &\leq x_i - t_i \leq x_i - r_i < \frac{1}{n(\lambda_1)}, \quad 1 \leq i \leq d, \\ \beta_S(\mathbf{v})_i - x_i &< t_i + \frac{1}{n(\lambda_1)} - x_i = r_i + \frac{1}{n(\lambda_1)} - x_i \leq \frac{1}{n(\lambda_1)}, \quad i \notin \{\ell, d+1\}, \\ \beta_S(\mathbf{v})_\ell - x_\ell &= t_\ell - x_\ell = r_\ell + \frac{1}{n(\lambda_1)} - x_\ell < \frac{1}{n(\lambda_1)}. \end{aligned}$$

Clearly, $x_{d+1} - \beta_S(\mathbf{v})_{d+1} < \frac{1}{n(\lambda_1)}$ or $\beta_S(\mathbf{v})_{d+1} - x_{d+1} < \frac{1}{n(\lambda_1)}$. Let \mathbf{z} be the lattice point in S such that $\beta_S(\mathbf{z}) = \mathbf{x}$. Then \mathbf{z} and \mathbf{v} contradict Lemma 3.1, and thus we have $\beta_S(S \cap \mathbb{Z}^d) = R_{\lambda_1}$.

We now show that S and $\lambda_1^{-1}S_d$ are unimodularly equivalent. Let $S = \text{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$. Since $\beta_S(S \cap \mathbb{Z}^d) = R_{\lambda_1}$ there are exactly $n(\lambda_1) + 1$ lattice points on every edge of S . Therefore $\mathbf{w}_i = \frac{1}{n(\lambda_1)}(\mathbf{v}_i - \mathbf{v}_{d+1}) \in \mathbb{Z}^d$ for all i . Moreover, by Lemma 5 in [24] it holds $\text{vol}(\lambda_1 S) \leq \text{vol}(S_d)$ and letting M be the $d \times d$ Matrix having columns $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d$ we conclude

$$|M| = \frac{d!}{n(\lambda_1)^d} \text{vol}(S) \leq \frac{d!}{n(\lambda_1)^d} \text{vol}(\lambda_1^{-1}S_d) = \frac{d!}{(d+1)^d} \text{vol}(S_d) = 1.$$

Thus, M is unimodular and the equation

$$M(\lambda_1^{-1}S_d + \lambda_1^{-1}\mathbf{1}) + \mathbf{v}_{d+1} = S,$$

implies that S and $\lambda_1^{-1}S_d$ are indeed unimodularly equivalent. \square

4. THE PROOF OF THEOREM 1.2

Let $\text{bd}(K)$ denote the boundary of a set K . Pick proved the following remarkable identity for lattice polygons.

Theorem 4.1 (Pick, [22]). *Let P be a lattice polygon in the plane. Then*

$$G(P) = \text{vol}(P) + \frac{1}{2}G(\text{bd}(P)) + 1.$$

Pick's theorem has been generalised to higher dimensions by Ehrhart, cf. [9–11], and for a modern introduction to the underlying theory we refer to [3]. Scott [25] stated the following result having a very similar flavor.

Theorem 4.2 (Scott, [25]). *Let P be a convex lattice polygon with at least one interior point. Then*

$$G(\text{bd}(P)) - 2G(\text{int}(P)) \leq 7,$$

and equality is attained if and only if P is unimodularly equivalent to S_2 .

One of the most fascinating theorems regarding convex bodies and their centroids was presented by Grünbaum. It provides an interesting property of the centroid in terms of mass-distribution of the given convex body.

Theorem 4.3 (Grünbaum, [15]). *Let $K \in \mathcal{K}_c^d$ be convex body and let $H \subset \mathbb{R}^d$ be a half-space containing the centroid $c(K)$, then*

$$\text{vol}(K \cap H) \geq \left(\frac{d}{d+1} \right)^d \text{vol}(K).$$

It is noteworthy that equality holds in Grünbaum's theorem if K is a simplex.

In order to prove Theorem 1.2 we will also use the following theorem by Ehrhart discussed already in Section 1.

Theorem 4.4 (Ehrhart, [8, 13]). *Let $K \in \mathcal{K}_c^2$ with $\text{vol}(K) \geq 9/2$. Then K contains at least two lattice points distinct from the origin.*

Proof of Theorem 1.2. Let $P = \text{conv}(K \cap \mathbb{Z}^2)$. If $\mathbf{0} \notin \text{int}(P)$ then there exists a half-space H containing P and containing $\mathbf{0}$ in its boundary such that $K \cap H \cap \mathbb{Z}^2 = K \cap \mathbb{Z}^2$. Grünbaum's Theorem and Ehrhart's Theorem imply that

$$\text{vol}(P) \leq \text{vol}(K \cap H) \leq \left(1 - \left(\frac{2}{3} \right)^2 \right) \text{vol}(K) = \frac{5}{9} \text{vol}(K) \leq \frac{5}{2}.$$

In turn applying Pick's Theorem yields

$$G(K) = G(K \cap H) = G(P) = \text{vol}(P) + \frac{1}{2}G(\text{bd}(P)) + 1 \leq \frac{7}{2} + \frac{1}{2}G(K).$$

Thus $G(K) \leq 7$.

Next we assume thus that $\mathbf{0} \in \text{int}(P)$. Applying the theorems of Ehrhart and Pick again gives that

$$\begin{aligned} \frac{9}{2} &\geq \text{vol}(K) \geq \text{vol}(P) = G(P) - \frac{1}{2}G(\text{bd}(P)) - 1 \\ &= G(K) - \frac{1}{2}(G(K) - 1) - 1 = \frac{1}{2}G(K) - \frac{1}{2}, \end{aligned} \tag{4.1}$$

and thus $G(K) \leq 10$.

Now, let $G(K) = 10$. Then (4.1) implies that $\text{vol}(K) = \text{vol}(P)$ and thus $P = K$ by compactness. Furthermore, we know that P contains a lattice point in its interior as $G(P) = 10$. Scott's Theorem implicates that $P = K$ is unimodularly equivalent to S_2 . \square

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